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A theory of thermal oscillations in liquid metals

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Thermal oscillations have been found to occur during crystal growing, thereby introducing undesirable striations in the solid crystal produced. A reason for the existence of such oscillations is given in this paper, and relevant experiments discussed.

1. Introduction

In the process of growing metal and semi-conductor crystals from a liquid melt, spontaneous oscillations in the temperature of the melt have been found to occur, thereby giving undesirable properties to the solid crystal produced (Hurle 1967; Jakeman & Hurle 1972). Experiments (Hurle 1966; Skafel 1972; Hurle, Jakeman & Johnson 1974) have shown that such thermal oscillations occur in liquid metals when *a horizontal* temperature gradient is imposed, and only occur when this gradient is above a certain value. This suggests an instability of the basic state to disturbances of an osoillatory character. Hart (1972) has found such an instability for a basic state which is an exact solution of the governing equations. It is suggested that this is the same type of instability as is found in the experiments. The objectives of this paper are (i) to give a physical description of the nature of the oscillations, (ii) to describe a method for finding approximate stability characteristics and (iii) to discuss the experiments in the light of the available theory. The theory cannot explain *all* the experimental results. However, it is felt that an adequate explanation is given of *why* oscillations should occur.

In order to describe the basic state, let (x, y, z) be a right-handed system of co-ordinates such that the z axis points vertically upwards and the x axis is in the direction of the horizontal temperature gradient ($T_x > 0$). The horizontal buoyancy gradient will generate vorticity leading to a shear U_z, and advection by the associated velocity field generates a vertical temperature gradient T_z. An exact solution of the governing equations has velocity

$$(U(z), 0, 0) \tag{1.1}$$

and temperature T of the form

$$T = xT_x + O(x), \tag{1.2}$$

where T_x is a constant, and U and Θ satisfy

$$vu, = \alpha g T_x, \quad \chi\Theta_{zz} = UT_x. \qquad (1.3), (1.4)$$

g is the acceleration due to gravity, α is the thermal expansion coefficient, ν the kinematic viscosity and χ the thermal diffusivity of the fluid. Thus U is a cubic in z and Θ a quintic, the coefficients depending on the boundary conditions applied at plane horizontal boundaries $z = \pm \frac{1}{2}d$. Hart (1972) considered the stability of this basic state for cases where the boundaries are (a) rigid conductors and (b) rigid insulators; and has given results for a large range of values of the Prandtl number

$$P = \nu/\chi. \tag{1.5}$$

For liquid metals, P is very small (e.g. 0·026 for mercury, 0·02 for gallium) and so only the results for very small P are relevant. For such values of P, the first form of instability to occur was a 'transverse' mode. This is not oscillatory in character and so will not be considered further except in a discussion of the experiments. The oscillatory or 'overstable' disturbances were longitudinal even modes (labelled L, E in Hart's figures), 'longitudinal' meaning disturbance motion independent of x and 'even' describing symmetry about $z = 0$. These are the type of modes which will be examined in this paper. The disturbance equations are given in § 2, approximate solutions are found in § 3 and the small-P limit discussed in §§ 4 and 5. The experiments are then discussed in § 6.

2. The perturbation equations

Consider a perturbation to the basic state defined by (1.1) and (1.2). As far as the perturbation equations are concerned, it is not necessary that (1.3) and (1.4) be satisfied by U and Θ, i.e. that the basic state be an exact solution of the equations. The perturbation equations will apply as long as (1.1) and (1.2) give a sufficiently accurate approximation to the basic state, which could well be true near the centre of a cavity whose depth is small compared with its length.

Let the perturbation velocity be (u, v, w) and the temperature perturbation be θ, where u, v, w and θ are functions of y, z and t only, i.e. the perturbation is *independent* of x but varies in the vertical plane which is perpendicular to that of the basic flow. (The idea of considering this type of perturbation came from the measurements of Hurle *et al.* (1974, figure 12). The lines of constant phase tend to be parallel to the flow.) A stream function ψ can be introduced to describe flow in this plane, with

$$v = -\psi_z, \quad w = \psi_y. \tag{2.1}$$

The three remaining equations are (a) the temperature equation

$$\theta_t + uT_x + \psi_y \Theta_z = \chi \Delta \theta, \tag{2.2}$$

(b) the x component of the momentum equation

$$u_t + \psi_y U_z = \nu \Delta u, \tag{2.3}$$

and (c) the equation for the x component $\Delta\psi$ of vorticity, namely

$$\Delta \psi_t = \alpha g \theta_y + \nu \Delta^2 \psi. \tag{2.4}$$

In all these equations, the operator Δ is defined by

$$\Delta \psi = \psi_{yy} + \psi_{zz}. \tag{2.5}$$

These equations may now be reduced to a single equation for ψ. Equation (2.4) gives θ in terms of ψ, and if this is substituted in the y derivative of (2.2), an expression for u in terms of ψ is obtained, namely

$$-\alpha g T_x u_y = \left(\frac{\partial}{\partial t} - \chi\Delta\right)\left(\frac{\partial}{\partial t} - \nu\Delta\right)\Delta\psi + \alpha g \Theta_z \psi_{yy}. \tag{2.6}$$

When this is substituted in the y derivative of (2.3), the result is

$$\left(\frac{\partial}{\partial t} - \nu\Delta\right)^2\left(\frac{\partial}{\partial t} - \chi\Delta\right)\Delta\psi + \left(\frac{\partial}{\partial t} - \nu\Delta\right)(\alpha g \Theta_z \psi_{yy}) = \alpha g T_x U_z \psi_{yy}. \tag{2.7}$$

Solutions of (2.7) may be found with the form

$$\psi = e^{\sigma t}\sin(ly)\,\hat{\psi}(z), \tag{2.8}$$

which, when substituted in (2.7), reduces it to an ordinary differential equation of order eight. The stability problem involves finding the eigenvalues σ for which the appropriate boundary conditions are satisfied. Instability occurs when there exists an eigenvalue σ with positive real part.

The stability characteristics depend on two non-dimensional numbers, the Prandtl number P defined by (1.4) and a Rayleigh number

$$A = \alpha g T_x d^4/\nu\chi \tag{2.9}$$

based on the horizontal temperature gradient T_x and depth d of fluid. The way these parameters enter the problem can be seen when (2.8) is put in non-dimensional form. Non-dimensional quantities are defined by

$$\left.\begin{aligned} z_* = z/d, \quad l_* = ld, \quad \sigma_* = \sigma d^2(\nu\chi)^{-\frac{1}{2}},\\ u_z = -d^2U_z/\chi A, \quad \tau_z = \Theta_z/AT_x. \end{aligned}\right\} \tag{2.10}$$

When (2.8) is substituted in (2.7) and use is made of (2.10), the result is

$$(\sigma_* - P^{\frac{1}{2}}\Delta_*)^2(\Delta_* - P^{\frac{1}{2}}\sigma_*)\Delta_*\hat{\psi} + l_*^2A^2[u_z\hat{\psi} + P^{\frac{1}{2}}(\sigma_* - P^{\frac{1}{2}}\Delta_*)(\tau_z\hat{\psi})] = 0, \tag{2.11}$$

where

$$\Delta_* = d^2/dz_*^2 - l_*^2. \tag{2.12}$$

Equation (2.11) is the governing equation of the problem.

3. Approximate solutions of the stability problem

Approximate solutions to (2.11) can be found by replacing u_z and τ_z by constants $\bar{u}_z$ and $\bar{\tau}_z$, which can be regarded as appropriately weighted mean values of u_z and τ_z. Such an approach has been used in an unpublished report by Faller (see Faller 1969). The agreement with the exact result will depend, to some extent, on the suitability of the weighting function used. At worst, the method can be regarded as a form of scale analysis which will show how σ depends on the given parameters in various limits. In practice, the method gives results which agree with Hart's numerical results and with experiments to within a factor of two.

If u_z and τ_z are constant, (2.11) has solutions which are sinusoidal in z. A vertical scale is set by assuming that the depth d of fluid will be approximately half a wavelength, and hence the operator Δ_* in (2.11) is replaced by

$$\Delta_* = -\pi^2 - l_*^2 = -k^2. \tag{3.1}$$

Then (2.11) becomes an algebraic equation

$$(\sigma_* + P^{\frac{1}{2}}k^2)^2 (k^2 + P^{\frac{1}{2}}\sigma_*) k^2 + l_*^2 A^2[\bar{u}_z + (P^{\frac{1}{2}}\sigma_* + Pk^2)\bar{\tau}_z] = 0. \tag{3.2}$$

For solutions with an oscillatory character, σ_* is complex, i.e.

$$\sigma_* = \sigma_r + i\sigma_i \tag{3.3}$$

with $\sigma_i \neq 0$. Substituting in (3.2) and taking real and imaginary parts, one obtains a cubic equation for σ_r and an expression for σ_i^2 in terms of σ_r and the other parameters. The conditions for marginal instability are obtained by putting $\sigma_r = 0$. They can be found directly from (3.2) by putting $\sigma_* = i\sigma_i$ in (3.2) and taking real and imaginary parts. Two expressions for σ_i^2 result, namely

$$\sigma_i^2 = \frac{l_*^2 A^2 \bar{u}_z + P(k^8 + k^2 l_*^2 A^2 \bar{\tau}_z)}{(1+2P)k^8} = \frac{(2+P)k^6 + l_*^2 A^2 \bar{\tau}_z}{k^2}. \tag{3.4}$$

These expressions combine to give $A = A_c$ as the condition for marginal stability, where

$$A_c^2 = \frac{2(1+P)^2 k^8}{l_*^2[\bar{u}_z - \bar{\tau}_z(1+P)k^2]}. \tag{3.5}$$

Unstable solutions are found for $A > A_c$, and stable ones for $A < A_c$. Oscillatory instability is only possible when the right-hand side of (3.5) is positive, i.e.

$$\bar{\tau}_z(1+P)k^2 < \bar{u}_z. \tag{3.6}$$

To obtain more definite expressions, the weighting factor $\cos^2 \pi z_*$ was used for the solutions of (1.3) and (1.4). For the case of rigid conducting boundaries ($U = \Theta = 0$ at $z = \pm\frac{1}{2}d$), this gives

$$\bar{u}_z = (2\pi)^{-2}, \quad \bar{\tau}_z = (2\pi)^{-4}, \tag{3.7}$$

while in the case of free conducting boundaries

$$\bar{u}_z = (2\pi)^{-2}(1+\tfrac{1}{3}\pi^2), \quad \bar{\tau}_z = (2\pi)^{-4}(1+\tfrac{1}{3}\pi^2). \tag{3.8}$$

In either case (3.6) and (3.1) show that instability is only possible when $P < 3$. (The number 3 is, of course, only the approximate value given by the model.) For larger P, it can be seen from (3.5) that the stabilizing effect of the vertical temperature gradient becomes too great for instability to occur.

For small P, the value of l_*^2 that gives the smallest value of A_c in (3.5) corresponds, in both cases, to a wavelength λ in the y direction and a frequency σ_i given by

$$\lambda = 3{\cdot}7d, \quad \sigma_i = 22. \tag{3.9}$$

The corresponding value of A_c is

$$A_c = 1030 \tag{3.10}$$

for rigid conducting boundaries and

$$A_c = 240 \tag{3.11}$$

for free conducting boundaries. The results for rigid conducting boundaries may be compared with the exact values found by Hart (1972), namely $\lambda = 3{\cdot}1d$, $A_c = 1700$ and $\sigma_i = 40$. All the approximate values are within a factor of two.

4. The small-P limit

The results of the previous section for constant u_z and τ_z and a fixed vertical wavenumber show that A_c and σ are of order unity as P tends to zero. Assuming this to be true in the general case, the small-P limit can be investigated and a physical description of the mechanism responsible for the oscillations can be obtained.

Suppose that, as $P \to 0$, $\hat{\psi} \to \psi_0$ and $\sigma_* \to \sigma_0$. Then (2.11) formally gives

$$\sigma_0^2 \Delta_*^2 \psi_0 + l_*^2 A^2 u_z \psi_0 = 0. \tag{4.1}$$

This limit is valid in an interior region, but cannot be uniformly valid over the whole domain for all boundary conditions since (4.1) is of fourth order whereas the full equation (2.11) is of order eight. In order to find the physical meaning of this equation, and to see what boundary conditions are appropriate, one can look at the corresponding limiting forms of the original equations (2.2)–(2.4). These are

$$uT_x = \chi \Delta\theta, \tag{4.2}$$

$$u_t + \psi_y U_z = 0, \tag{4.3}$$

$$\Delta\psi_t = \alpha g \theta_y, \tag{4.4}$$

i.e. *diffusion dominates* in the heat equation but *viscous effects are negligible*. The boundary condition on momentum, therefore, is the inviscid one of no normal flow, i.e.

$$\psi_0 = 0, \tag{4.5}$$

while the condition on temperature is $\theta = 0$ for the conducting case, i.e. by (4.4),

$$\Delta_* \psi_0 = 0. \tag{4.6}$$

Alternatively, in the insulating case, it is $\theta' = 0$, i.e.

$$\Delta_* \psi_0' = 0, \tag{4.7}$$

where the prime denotes a derivative with respect to z_*.

For free conducting boundaries, ψ_0 satisfies the full viscous boundary conditions, and so is a uniformly valid first approximation to the full solution. If (4.1) is multiplied by ψ_0 and integrated over the depth, there results

$$\sigma_0^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} (\psi_0''^2 + 2l_*^2 \psi_0'^2 + l_*^4 \psi_0^2)\, dz_* = -l_*^2 A^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} u_z \psi_0^2 \, dz_*, \tag{4.8}$$

showing that σ_0^2 must be negative, since u_z is positive over the entire depth in this case. A similar result may be obtained by multiplying (4.1) by $\Delta_* \psi_0$ and

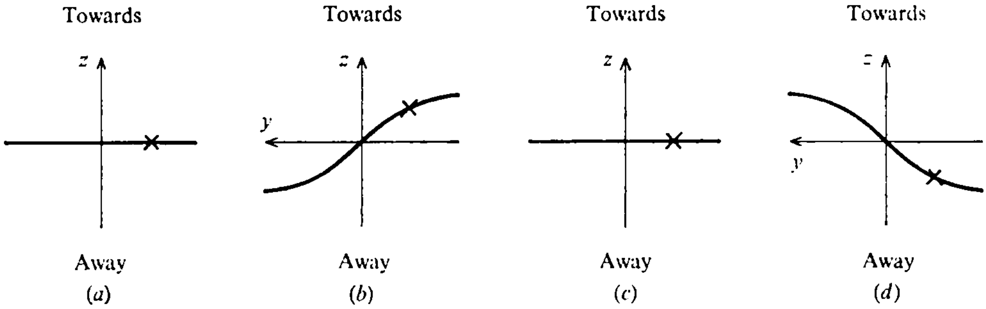

FIGURE 1. Successive positions (thick line) of a line of particles undergoing oscillations. Their equilibrium position is $z = 0$ and the x axis points into the page. The fluid above $z = 0$ is moving towards the reader and towards the cold wall. The fluid below $z = 0$ is moving away from the reader. The signs of w and θ are exactly the same as for an internal gravity wave, but the way the temperature perturbation θ is determined is entirely different. For zero Prandtl number, the motion is inviscid but heat diffuses instantaneously. Since x momentum is conserved the perturbation velocity u is positive, i.e. away from the cold wall, when the particle is elevated. Advection away from the cold wall produces a negative temperature perturbation. Since diffusion is instantaneous, the minimum value of θ occurs when the particle elevation is a maximum, and a simple harmonic oscillation results.

For small non-zero Prandtl number, diffusion is not quite instantaneous and the particle reaches its minimum temperature a little after it reaches its maximum elevation. This will lead to growth of the disturbance if the phase shift is sufficient to overcome the effects of viscous damping. The condition for this to be achieved is that the Rayleigh number A based on the horizontal temperature gradient T_x and the depth d be above a certain value.

(*a*) $w > 0, u = 0, \theta = 0$. (*b*) $w = 0, u > 0, \theta < 0$. (*c*) $w < 0, u = 0, \theta = 0$. (*d*) $w = 0$, $u < 0, \theta > 0$.

integrating. Equation (4.8) may be used to find σ_0^2 by the Rayleigh–Ritz procedure. ψ_0 can be expanded as a Fourier cosine series, since these functions satisfy the boundary conditions. A first approximation is obtained by putting

$$\psi_0 = \cos \pi z_*,$$

which gives precisely the results given in § 3 for the free-boundary case. This can be regarded as a justification of the procedure used in § 3, at least for the case of free conducting boundaries.

The physical nature of the oscillation can now be described with the help of (4.2)–(4.4). Figure 1 shows four stages in the oscillation. The line drawn represents a line of particles whose equilibrium position is $z = 0$. The choice of axes is such that T_x is positive and U_z is negative, i.e. the fluid below the y axis is moving away from the reader and away from the cold wall, whereas the reverse is true above the y axis. The particle conserves its x momentum, so when it is elevated its perturbation velocity is away from the reader and away from the cold wall. Thus it is cold relative to its surroundings, and since diffusion takes place instantaneously in the limit considered, the particle is coldest when at its maximum elevation. The restoring force is therefore a maximum at this point, and since the fluid is effectively inviscid, the oscillation is maintained.

The frequency of oscillation is of order

$$\left(\frac{\alpha g T_x U_z}{\chi}\right)^{\frac{1}{2}} \frac{l}{l^2+m^2}, \tag{4.9}$$

where m is the effective vertical wavenumber.

Physical arguments can also be used to see how the neglected effects will modify the oscillation. First, there is the effect of the finite diffusion time, so that the particle will not cool down instantaneously as it is elevated and will reach its minimum temperature a little after its maximum elevation. At zero displacement, the descending particle will still be slightly colder than its surroundings and so there is a force tending to increase the amplitude of the oscillation. This effect is, therefore, destabilizing. On the other hand, there are two effects which tend to stabilize the motion, namely viscous dissipation and the presence of a stable vertical temperature gradient. Whether the oscillation will be self-excited or not therefore depends on which of these effects is most important. It will be found that the destabilizing effects become more important when the Rayleigh number is above a certain value.

5. Determination of marginal-stability conditions

The first approximation for small P determines the frequency of oscillation, but does not determine whether the oscillation is stable or unstable. The condition for marginal stability depends on the next order of approximation, i.e. it depends on a balance between small effects. Determination of the next approximation is most straightforward for the case of free conducting boundaries. In this case

$$\sigma_* = \sigma_0 + P^{\frac{1}{2}}\sigma_1 + \ldots, \tag{5.1}$$

and, in the interior at least,

$$\hat{\psi} = \psi_0 + P^{\frac{1}{2}}\psi_1 + \ldots, \tag{5.2}$$

where, by substitution in (2.11), ψ_1 satisfies

$$\sigma_0^2 \Delta_*^2 \psi_1 + l_*^2 A^2 u_z \psi_1 + 2\sigma_0(\sigma_1 - \Delta_*)\Delta_*^2 \psi_0 - \sigma_0^3 \Delta_* \psi_0 + \sigma_0 l_*^2 A^2 \tau_z \psi_0 = 0. \tag{5.3}$$

σ_1 is now determined by a solvability condition. Since the operator acting on ψ_1 in (5.3) is self-adjoint for free conducting boundaries, this condition is obtained by multiplying (5.3) by ψ_0 and integrating over the depth. This gives

$$\begin{aligned} 2\sigma_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} (\psi_0''^2 + 2l_*^2 \psi_0'^2 + l_*^4 \psi_0^2)\, dz_* &= -2\int_{-\frac{1}{2}}^{\frac{1}{2}} (\psi_0'''^2 + 3l_*^2 \psi_0''^2 + 3l_*^4 \psi_0'^2 + l_*^6 \psi_0^2)\, dz_* \\ &\quad - \sigma_0^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} (\psi_0'^2 + l_*^2 \psi_0^2)\, dz_* - l_*^2 A^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \tau_z \psi_0^2\, dz_*. \end{aligned} \tag{5.4}$$

The second term on the right-hand side is positive since σ_0^2 is negative, and represents the destabilizing effect due to the finite diffusion time. The preceding term represents viscous dissipation and the succeeding term, the stabilizing effect of the stable vertical temperature gradient. Approximate solutions can be found by using the functions used in determining σ_0 from (4.8) by the Rayleigh–Ritz procedure. If a cosine expansion is used, the first approximation obtained by substituting in (5.4) is that obtained already in § 3 for the free conducting case. This is considered to justify, in this case, the procedure used in § 3, and to demonstrate the possible existence of unstable oscillations in liquid metals.

For rigid boundaries, the above expansion is not valid. In this case, ψ_0 does not satisfy the full viscous boundary conditions, and so thin Stokes layers are required adjacent to the boundaries, in order to satisfy the no-slip conditions. These layers have thickness $(P/\sigma_i)^{\frac{1}{2}}$. According to Hart's (1972) results, this thickness is $\frac{1}{40}$ of the depth for $P = 0{\cdot}026$. The existence of the Stokes layer increases viscous dissipation by a factor of order $P^{-\frac{1}{2}}$, so to obtain marginal instability, A has to be increased by a factor of order $P^{-\frac{1}{8}}$ in order to increase the destabilizing effect of the finite diffusion time. This weak dependence on P is not detectable in Hart's results. The matter will not be pursued further in this paper, as the emphasis is on the nature of the phenomenon rather than on detailed results.

6. Discussion of experimental results

The experimental results of Skafel (1972) and Hurle *et al.* (1974) will now be discussed. The latter authors studied convection in a rectangular box containing molten gallium, the convection being driven by maintaining a temperature difference between the two ends. Skafel used mercury as the working fluid and studied convection both in the rectangular geometry and in an annulus with a temperature difference maintained between the inner and outer cylindrical surfaces. In Skafel's apparatus, the lower boundary was insulating and the upper surface was usually exposed to the air. In each case measurements were made of the mean temperature field and of thermal oscillations, and the smallest temperature difference for which oscillations occurred was ascertained for different geometrical parameters.

In the box geometry, no evidence was given which would indicate the presence of transverse modes (see Hart 1972) resulting from shear-flow instability. One concludes either that the side walls of the box stabilized this form of disturbance, or that the measurement techniques were not sensitive to its presence. The measured temperature fields indicated a single-cell circulation whereas transverse modes, if present, would perhaps result in secondary circulations. In the annular configuration, there are no side walls to inhibit the shear-flow instability. In this case, Skafel's measurements of the mean temperature field looked much the same as those for the box geometry, i.e. they indicated a single-cell circulation. However, he reported (Skafel 1972, p. 18) that "small amplitude fluctuations with a wide frequency bandwidth were present, the shapes of which were not consistent from sample to sample". Thermal oscillations, when they occurred, stood out clearly above this background noise. The small amplitude fluctuations may have been the result of shear-flow instability. However they appear to have been rather unimportant, and did not prevent the thermal oscillations from occurring.

Let us consider the results for the annular configuration first. Suppose that the radii of the outer and inner cylindrical walls are R_o and R_i respectively. Suppose that ΔT is the temperature difference between the two cylindrical walls and d the depth of fluid. Then the parameters on which the nature of the convection depends are the Prandtl number P, the radius ratio

$$R = R_o/R_i, \tag{6.1}$$

the aspect ratio

$$h = d/L, \tag{6.2}$$

where $L = R_o - R_i$, and the difference Rayleigh number

$$A_d = \alpha g \Delta T d^4/\nu\chi L. \tag{6.3}$$

The basic steady-state solution in this geometry is not known. For small P and small A, the temperature field will be determined purely by conduction, so that in the conducting-boundary case the horizontal temperature gradient would be approximately constant if the radius ratio were not too large. For small aspect ratio, the flow away from the ends would be approximately the cubic velocity profile satisfying (1.3).

However, the oscillatory instability does not arise until the Rayleigh number is of order 1000, by which time convective effects have become important in determining the temperature field. Measurements show the horizontal temperature gradient to be larger than $\Delta T/L$ near the end walls, but to be approximately linear over the central portion with a value smaller than $\Delta T/L$. This allows one to determine a gradient Rayleigh number

$$A_g = \alpha g T_x d^4/\nu\chi \tag{6.4}$$

based on the observed horizontal temperature gradient T_x in the central part of the cavity.

The vertical temperature gradient at the onset of thermal oscillations was also measured. For small depths of fluid this was negative because of surface cooling. For larger depths, the measured value T_z given is the average over the region in which T_z was positive. In table 1, the measured T_z is compared with the value given by (3.7) (using the measured T_x and A_g). T_z is seen to have the order of magnitude given by (3.7), but to vary somewhat in its numerical value. For that reason, critical conditions were calculated from (3.4) and (3.5) for a variety of values of T_z and the results are shown in table 2. Since $\bar{u}_z$ is unknown, this was kept fixed at the value given by (3.7).

The theory discussed in earlier sections cannot be expected to agree with experiment in any precise way because of lack of knowledge of the basic flow, approximations made in calculating stability characteristics, and the neglect of end effects. However it is of interest to calculate experimental values of the Rayleigh number and of the non-dimensional frequency σ_i to see if they have values comparable with those predicted by the theory available. Table 1 shows

R_o/R_i	d/L	A_d	σ_i	$(2\pi)^4\,\tau_z$	A_g	$\sigma_i/\sigma_{\text{th}}$	λ/d
1·38	0·30	610	10	0·9	410	0·5	—
	0·40	890	15	2·0	410	0·6	—
	0·47	1330	21	0·7	900	1·0	2·3
	0·50	1540	23	1·2	730	1·0	—
	0·55	2310	32	1·4	920	1·3	—
	0·60	3910	40	—	—	—	—
	0·70	4850	44	—	—	—	—
	0·80	6180	44	—	—	—	—
1·73	0·175	660	10	−2·2	400	0·7	—
	0·20	720	14	0·5	360	0·7	4·0
	0·25	860	19	2·6	520	0·6	—
	0·30	1050	20	0·5	560	1·0	—
2·49	0·125	850	—	−1·6	560	—	—
	0·15	960	14	−0·4	530	0·8	—
	0·175	1220	18	1·1	800	0·8	—
	0·2	1490	20	0·9	1130	0·9	2·8
	0·25	2420	29	—	—	—	—

TABLE 1. Results for marginal stability for mercury ($P = 0{\cdot}026$) in an annulus (from Skafel 1972). R_o is the radius of the outer cylinder and R_i the radius of the inner cylinder. $L = R_o - R_i$ and d is the depth. A_d is the difference Rayleigh number based on $\Delta T/L$ and the depth. $\sigma_i = d^2\omega(\nu\chi)^{-\frac{1}{2}}$, where ω is the angular frequency. A_d and σ_i are taken from Skafel's table 1. A_g is the gradient Rayleigh number based on the observed gradient T_x at the centre of the cavity (from Skafel's figures 9–11). τ_z is a non-dimensional vertical temperature gradient, depending on the observed vertical gradient and observed horizontal gradient at the centre of the cavity. σ_{th} is the theoretical frequency for the observed τ_z according to table 2. λ is the observed wavelength in the azimuthal direction (see text). Values of A_g and λ can be compared with those given in table 2.

$(2\pi)^4\,\bar{\tau}_z$	A_c	σ_i	λ/d
−3	600	14	3·2
−2	650	15	3·2
−1	730	16	3·3
0	850	19	3·5
1	1030	22	3·7
2	1340	27	4·3
3	2490	39	5·8

TABLE 2. Computed characteristics of the first marginally stable disturbance for different values of $\bar{\tau}_z$. A_c is the minimum value of A_c given by (3.5) and λ/d corresponds to the value of l_* which makes A_c a minimum. σ_i is then calculated from (3.4).

measured critical values of A_d and σ_i for Skafel's experiments, and values of A_g and λ/d where available. The values of A_d and σ_i were calculated from Skafel's table 1, and the values of λ/d from the text. The value of T_z was taken from table 3 of Skafel and the value of T_x from graphs of T as a function of x.

It will be noticed that A_d increases with d/L and varies by a factor of ten over the experimental range. However much of this change is due to changes in the basic state with d/L. First, the horizontal temperature gradient changes, and if measured values of T_x are used, it is found that A_g changes by a much smaller

w/d	d/L	A_d	σ_i	$(2\pi)^4\tau_z$	A_g	σ_i/σ_{th}
			Mercury			
6·4	0·15	780	9	−2·7	420	0·7
4·8	0·20	570	10	1·8	360	0·4
3·8	0·25	950	13	1·2	590	0·6
3·2	0·30	1220	16	1·9	650	0·6
2·7	0·35	2150	24	2·4	900	0·8
1·9	0·40	7320	79	1·3	2000	3·4
			Gallium			
1·1	0·35	3500	55			
1·1	0·40	4600	65			
1·1	0·52	9100	96			
2·0	0·21	500	16			
1·5	0·29	1400	35			
1·0	0·45	7000	83			
1·1	0·40	4600	65			
0·8	0·40	3800	55			
0·5	0·40	—	184			

TABLE 3. Results for marginal stability for mercury and for molten gallium for a rectangular cavity of width w and length L. d is the depth of fluid. The results for mercury are taken from table 1 and figure 12 of Skafel (1972). The results for gallium are taken from figures 5–8 and §3.1.3 of Hurle *et al.* (1974).

factor. Second, it is observed that the *vertical* temperature gradient also changes. The changes in A_g observed are of the same extent as is predicted in table 2 for the observed range of values of T_z. Considering the nature of the theory used for comparison, the agreement is remarkable and appears to confirm that the basic ideas developed in earlier sections are applicable to the experiments.

Experimental results for the box geometry are shown in table 3. The parameters in this case are P, A_d, h and the ratio w/d of width to depth. The results for mercury, where w/d was usually large, agree with theory just as well as the results for the annulus, except for the smallest value of w/d, i.e. 1·9. The results for gallium are all for $w/d < 2$, making the width less than half of optimum wavelength in all but one case. If it is assumed that the disturbance wavelength is half the width instead of the optimum value, then (3.4), (3.5) and (3.7) give $A_c = 1700$ and $\sigma_i = 42$ for $w/d = 1$ and $A_c = 2200$ and $\sigma_i = 55$ for $w/d = 0{\cdot}8$. Both A_c and σ_i become infinite when $w/d = 0{\cdot}58$, and according to § 3, oscillations are not possible for smaller values of w/d. This explains the trend in the results for variations in w/d, but cannot explain the systematic dependence on the length of the cavity, since the theoretical results are independent of the length. This dependence could be due to changes in the basic state with length, or to changes in the perturbation, or to a combination of these, and it is beyond the scope of this paper to investigate this effect.

7. Conclusions

The temperature oscillations found in the experiments for annular geometry and for *wide* rectangular cavities have the physical characteristics described in § 4, and arise spontaneously for the reasons stated in § 4. The oscillations are primarily longitudinal and may be explained in terms of a diffusion-dominated inviscid model. The buoyancy restoring force results from horizontal advection in the presence of a mean horizontal temperature gradient, and would be in phase with particle elevations if diffusion were instantaneous. The phase shift resulting from the finite diffusion time can make the oscillation grow, provided that the Rayleigh number, as defined in § 2, is large enough for viscous dissipation to be overcome. The experiments in *narrow* cavities show the oscillations to be modified by the geometrical constraints. Presumably the reason for their existence is basically the same, but considerable further effort would be needed to determine their characteristics.

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